Causality and Quantum Theory

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Abstract

In our miniproject, based on a thought experiment, we found certain inequalities, known as Bell's inequalities. Bell's inequalities can be used to verify causality. We expected that causality is never violated in nature. We looked at what quantum theory predicts and found that the predicted results violate Bell's inequalities.

1 Introduction

At first we should define what causality is. The definition says that 'causality denotes a necessary relationship between one event (called cause) and another event (called effect) which is the direct consequence of the first'. Let's have a look at how causality works in a particular thought experiment.

2 Thought Experiment

We imagine a situation where there are three people: Alice and Bob, who can't communicate with each other, and Cyril, who can communicate with both of them. (Figure 1)

Someone prepares envelope (Figure 2) for Cyril. It is one big envelope in which are 2 medium envelopes and in each of the medium envelopes are 3 small envelopes. These are numbered from 1 to 3 and each of them contains either a red (R) or a blue (B) disc.

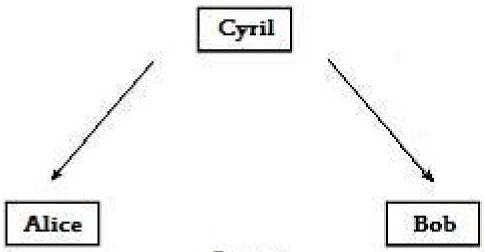

Figure 1

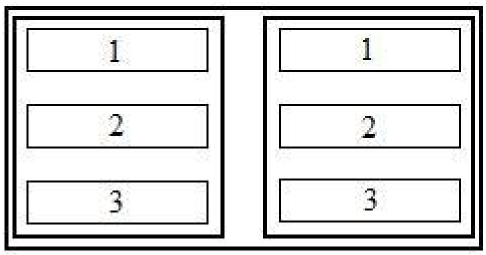

Figure 2

Cyril opens the big envelope and sends one of the medium envelopes to Bob and the other to Alice. Alice opens the envelope she has and chooses one of the 3 small envelopes, discarding the others. Now, Alice opens the small envelope she has chosen and checks whether there is red or blue disc. Bob does the same.

They keep repeating this experiment. Each time they write down number of the small envelope and color of the disc and thus they create a chart. Such as this one for example:

Alice	Bob
1 R	3 R
2 B	1 B
...	...

Now for each of the results there is certain probability of occurrence, which can be estimated with high accuracy if the experiment is repeated many times. Let these probabilities be denoted following the pattern, that the probability that Alice chose to open the first envelope and found a red disc inside, and that Bob opened the second envelope and found a blue disc is $p(1\mathrm{R}, 2\mathrm{B})=0$.

Let's now repeat the same experiment but with the envelopes prepared in given way. Now there is a condition that when we compare Alice's and Bob's medium envelopes, in small envelopes marked with same number can't be disc of same color. In other words, small envelopes of same number always contain different color.

Now we see that there are combinations which cannot appear in the chart Alice and Bob are making. Probability of these combinations is 0. Let's write these combinations down:

$$p(1\mathrm{R}, 1\mathrm{R})=0 \quad p(2\mathrm{R}, 2\mathrm{R})=0 \quad p(3\mathrm{R}, 3\mathrm{R})=0$$
$$p(1\mathrm{B}, 1\mathrm{B})=0 \quad p(2\mathrm{B}, 2\mathrm{B})=0 \quad p(3\mathrm{B}, 3\mathrm{B})=0$$

As well, based on the condition (different colors for same numbered small envelopes), the number of options how the envelopes can be prepared is limited. We create a table to note down all the possible combinations:

Type of large envelope	1st medium envelope	2nd medium envelope
a	1R, 2R, 3R	1B, 2B, 3B
b	1R, 2R, 3B	1B, 2B, 3R
c	1R, 2B, 3B	1B, 2R, 3R
d	1R, 2B, 3R	1B, 2R, 3B

As we can see, there are 4 'types' of large envelopes that can be prepared under the set condition. For each type of the envelopes goes different probability of its occurrence:

w_a for a type; w_b for b type; w_c for c type; w_d for d type.

We mark N as total number of big envelopes Alice and Bob open. Number of envelopes which fit for certain type of large envelope can be counted like this:

$$N_a=w_a \cdot N \quad (1) \qquad N_c=w_c \cdot N \quad (3)$$
$$N_b=w_b \cdot N \quad (2) \qquad N_d=w_d \cdot N \quad (4)$$

Because there are no other possibilities, it must hold that

$$N_a+N_b+N_c+N_d=N \qquad (5)$$
$$w_a \cdot N+w_b \cdot N+w_c \cdot N+w_d \cdot N=N \quad \text{... divided by N:}$$
$$w_a+w_b+w_c+w_d=1 \qquad (6)$$

as well it's obvious that $w_a, w_b, w_c, w_d \geq 0$

and therefore $w_a, w_b, w_c, w_d \leq 1$

Now let's have a look at probabilities of couple of combinations which Alice and Bob might get. We use the previous table for this.

$$P(1\mathrm{R}, 2R)=\left(\frac{1}{3}\right)^2 \cdot (w_c+w_d) \qquad (7)$$

$$P(2\mathrm{R},3R)=(\frac{1}{3})^{2}\cdot(w_b+w_d) \qquad (8)$$

$$P(1\mathrm{R},3R)=(\frac{1}{3})^{2}\cdot(w_b+w_c) \qquad (9)$$

It should hold that

$$P(1\mathrm{R},2\mathrm{R})+P(2\mathrm{R},3\mathrm{R})\geq P(1\mathrm{R},3R) \qquad (10)$$

, which is one of so-called Bell's inequalities [1]

We check if the inequality is valid by substituting from (7), (8) and (9) in (10):

$$(\frac{1}{3})^{2}\cdot(w_c+w_d)+(\frac{1}{3})^{2}\cdot(w_b+w_d)\geq(\frac{1}{3})^{2}\cdot(w_b+w_c)$$

We simplify the inequality to:

$$w_c+w_d+w_b+w_d\geq w_b+w_c$$

$$w_d+w_d\geq 0$$

which is always true, so we proved that the inequality holds for our thought experiment.

Next, we studied the validity of Bell's inequalities for completely randomly prepared large envelopes. In this case the probability of each event can be calculated in the following way. Let the quantity we are interested in be the probability of (1R, 2R). We can imagine this situation as on figure 3. Firstly we counted how many combinations can be substituted instead of question marks and with the knowledge of total number of possibilities we counted this probability.

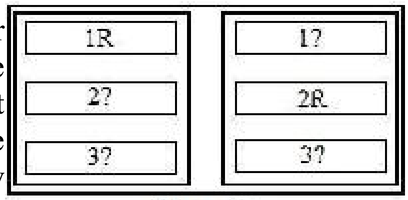

Figure 3

$$P(1\mathrm{R},2\mathrm{R})=\frac{16}{64}\cdot\frac{1}{9}\approx 0{,}027$$

Then we also tried to count probabilities for other combinations of discs and we found out that for each combination the probabilities are the same. Thus we expect that all of Bell's inequalities are true.

3 Violation of Bell's inequalities

Now, let's think about another experiment. In this experiment Alice and Bob measure spin of electrons (which can only be either plus or minus – equivalent of red and blue disks). There is pair of electrons in some special state (not to be described here) and Alice measures one of them and Bob the second one. In addition to this, they also choose angle of the measuring device (equivalent of numbers on small envelopes). For us will be enough to choose three fix vectors of measurements (which determine us the angles). Let's denote these vectors by $\vec{n}_1, \vec{n}_2, \vec{n}_3$.

Quantum theory tells us, that in an arrangement in which Alice is measuring the spin of her electron along the axis $\vec{n}$ and Bob is measuring his electron along the axis $\vec{n}\,'$, the probability that they obtain the same direction of spins is $q(\vec{n},\vec{n}\,')=\frac{1}{2}\cdot(1-\cos\vartheta)$ (11), where ϑ is angle enclosed by vectors $\vec{n},\vec{n}\,'$. Now we use this probabilities in Bell's inequality. First we need to write down the Bell's inequality appropriate for our case:

$$q(\vec{n}_1,\vec{n}_2)+q(\vec{n}_2,\vec{n}_3)\geq q(\vec{n}_1,\vec{n}_3)$$

Now we substitute in the Bell's inequality from (11)

$$\frac{1}{2}\cdot(1-\cos\vartheta_{12})+\frac{1}{2}\cdot(1-\cos\vartheta_{23})\geq\frac{1}{2}\cdot(1-\cos\vartheta_{13}) \quad (12)$$

where ϑ_{12} is angle between vectors $\vec{n}_{1,}\vec{n}_2$

ϑ_{23} is angle between vectors $\vec{n}_{2,}\vec{n}_3$

ϑ_{13} is angle between vectors $\vec{n}_{1,}\vec{n}_3$

And it also holds that

$$\vartheta_{12}+\vartheta_{23}=\vartheta_{13} \quad (13)$$

We continue solving the inequality (12):

$$1\geq-\cos\vartheta_{13}+\cos\vartheta_{12}+\cos\vartheta_{23} \quad (14)$$

We can substitute for ϑ_{13} from (13) and we get an inequality with two variables

$$1\geq-\cos(\vartheta_{12}+\vartheta_{23})+\cos\vartheta_{12}+\cos\vartheta_{23} \quad (15)$$

We find the maximum of this function at

$$\vartheta_{12}=\vartheta_{23}=\frac{\pi}{4} \quad .$$

Now, when we substitute this value back in (15), we see that the inequality doesn't hold:

$$1\geq-\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right)+\cos\frac{\pi}{4}+\cos\frac{\pi}{4}$$

$1\geq\sqrt{2}$, which is never valid

Which means that this Bell's inequality is violated. Since all causal processes should follow every Bell's inequality we conclude that quantum theory violates causality.

4 Conclusion

We have shown that causality does not work all the time and that it cannot explain all events that occur in nature. With use of Bell's inequalities we can see mathematical proof of this along with calculated example of situation when Bell's inequalities and therefore causality don't hold. We mention that experiments similar to that studied in Section 3 can actually be carried out, thus the causality of quantum theory can be verified, as it has been reported, for the first time in Ref. [2].

Saying thanks

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References

[1] John S. Bell, Speakable and Unspeakable in Quantum Mechanics, Cambridge University Press (1988)

[2] Alain Aspect, Jean Dalibard, and Géard Roger, Experimental Test of Realistic Local Theories via Bell's Theorem, Physical Review Letters, vol. 47 pp. 460, (1981)