Causality and Quantum Theory

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Abstract

Our miniproject studied the principles of causality and their relation to Bell's inequalities. We were introduced into the problem of causality and proved that the Bell's inequalities are not violated in causal systems. Then we proved that some Bell's inequalities may be violated in non-causal systems, simulated some of the systems where this occurs and found maximum possible violations.

1 Introduction

The casual systems are systems in which some events are indirectly determined by some other events. This plays an important role in Quantum mechanics.

2 Thought experiment

As a model of causal system, we imagined a thought experiment. Assume that we have three people (A,B,C) sitting in three rooms.
Certain rules apply:
C can comunicate with A and B and A cannot comunicate with B.
C receives a prepared envelope. In this large envelope there are two medium-sized envelopes. Within each medium-sized envelope, there are three small envelopes.
There is a red or blue disc in each of these small envelopes. C takes the large envelope and randomly distributes the medium-sized envelopes to A and B. A and B open the envelopes and randomly pick one of the small envelopes. Then they look at the disc inside and write down the color.

Figure 2 shows the example of what they may have written down.

A	B
1R	2B
3R	3B
1B	1R
2B	1R
2R	3B
1B	1R
3R	2R

Figure 1: Possible results

As you can see there is one important rule: C only puts discs of opposite colors in the envelopes denoted by the same number. (The two small envelopes denoted by the same number coming from a single large envelope are *anticorrelated*.)
This means that when A opens the first envelope, he knows what is in B's envelope of the same number.

According to this rule there exist only four different types of large envelopes (figure 1)
We will call these four types of envelopes a, b, c and d.

The evelopes are chosen randomly, with their respective probabilities W_a, W_b, W_c and W_d.

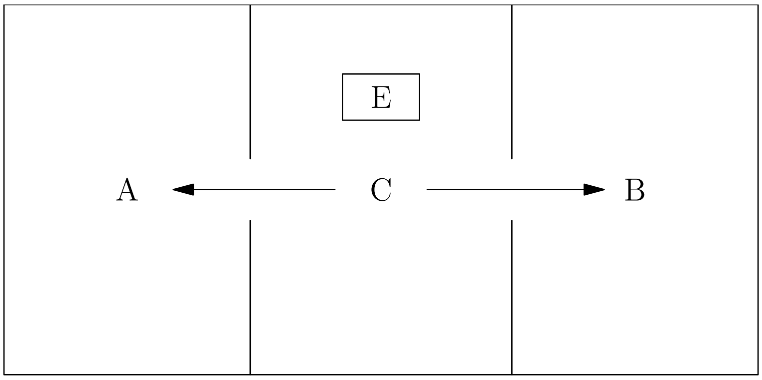

Figure 2: Thought experiment scheme

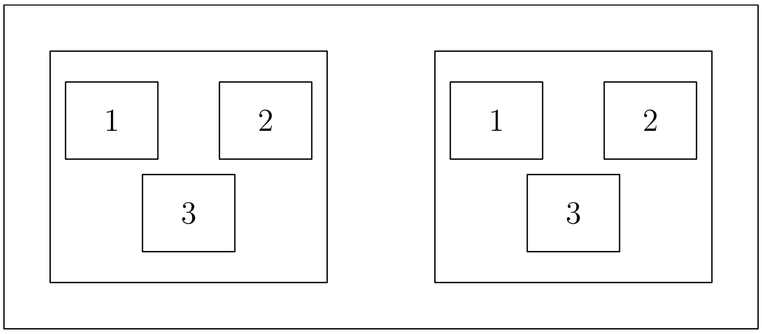

Figure 3: A large envelope

The probability of A choosing the first envelope and finding a red disc and B choosing the first envelope and finding a red disc is zero.

$$P(1R, 1R) = 0$$

the probability of A choosing the second envelope and finding a red disc and B choosing the second envelope and finding a blue disc is given by this formula:

$$P(2R, 2B) = \frac{1}{3^2}\frac{1}{2}$$

because there are three small envelopes to pick from.
The probability of A choosing 1B and B choosing 3R is:

$$P(1B, 3R) = \frac{1}{3^2}(W_c + W_d)\frac{1}{2}, \tag{1}$$

where and similar formulas for the other probabilites.

There we can get to the Bell's inequalities. One of them says that

$$P(1R, 2R) + P(2R, 3R) \geq P(1R, 3R)$$

Which leads to:

$$\frac{1}{9}(W_c + W_d)\frac{1}{2} + \frac{1}{9}(W_b + W_d)\frac{1}{2} \geq \frac{1}{9}(W_b + W_c)\frac{1}{2}$$

Simplified:

$$W_d \geq 0$$

Which always holds, because W_d is a probability. This proves that Bell's inequality in Eq. (1) is always satisfied.

This would be true for any general choice of large envelopes (not only in this case with opposite colors).
There are many other Bell's inequalities.

W_a	R, R, R	B, B, B
W_b	R, R, B	B, B, R
W_c	R, B, R	B, R, B
W_d	R, B, B	B, R, R

Figure 4: The four types of envelopes

3 Simulation

In the simulation, it was our task to numerically verify that Bell's inequalities are valid. We programmed a mathematical model in C++ which generated random envelopes with certain degrees of anticorrelation. The degree of anticorrelation was given by the factor ε.
If the $\varepsilon = 0$, the two medium envelopes are anticorrelated the most.
If the $\varepsilon = 1$, the two medium envelopes are independent and totally random.

To verify the Bell's inequality, we computed values of the quantity:

$$f = P(1R, 2R) + P(2R, 3R) - P(1R, 3R),$$

which should be non-negative when the inequality holds and negative when it is violated.

Figure 5 shows the results from our simulation.

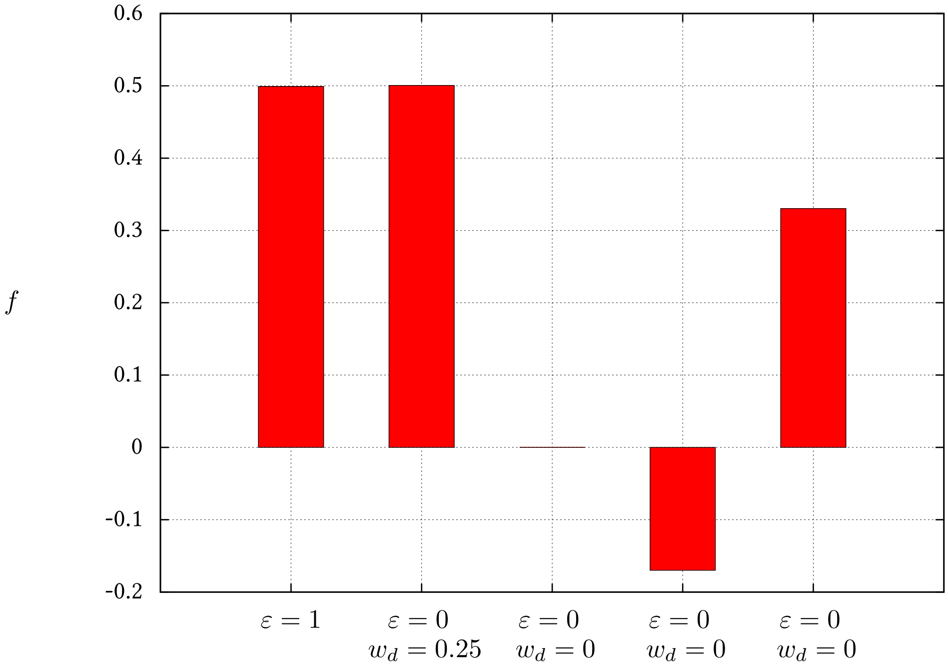

Figure 5: Results of the simulation. Negative values indicate violation of Bell's inequality.

We have obtained the first three bars, complying with the above rules. Then we have introduced a violation of causality with the following rule. When A opens the first envelope and finds a blue disc, then the disc in the third envelope of B turns blue.
The fourth and the fifth bars correspond to this case.
The Bell's inequality we used detects the violation only in the case when $W_a = W_b = W_c = \frac{1}{3}$ and $W_d = 0$. The violation in case of equal probabilities is undetected, and we would need to use a different Bell's inequality to detect it.

4 Violation of Bell's inequalities by Quantum Mechanics

We can move this experiment into the quantum domain, where we observe quantum particles instead of colored discs. This experiment is called The Bell Test of Quantum Mechanics.
The setup of this experiment is fundamentally the same as the first thought experiment's.

Instead of the medium-sized envelopes, we use a pair of correlated electrons, send them to A and B and let them measure their spins. A's and B's measuring apparatuses consist of three pairs of magnets shown in figure 6. A and B can decide which set of magnets they use for measuring the electron's spin. This corresponds to picking a small envelope out of the medium envelope. The spin of the electron can be measured in any direction by a pair of magnets. and can take the values of $\pm\frac{\hbar}{2}$. The correlations between the two electrons is such that, when their spins are measured in the same direction, they yield opposite results.

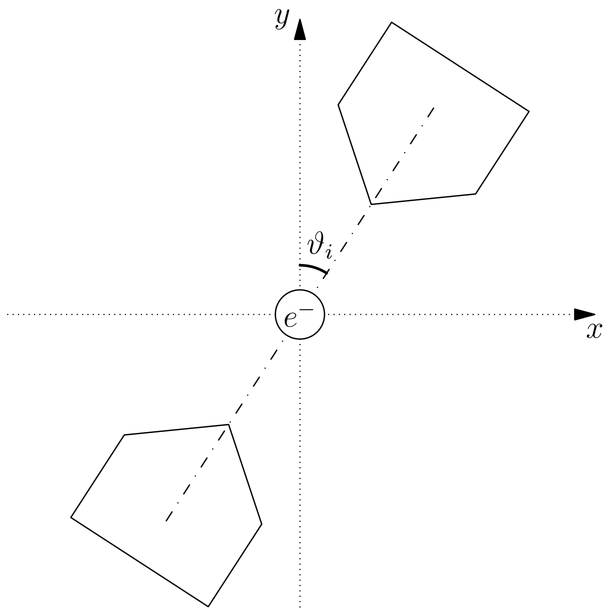

Figure 6: Spin analyzing apparatus

Let ϑ_i be the angle between a fixed axis and the axis of the i-th pair of magnets.
Then the difference between two angles is $\vartheta_1 - \vartheta_2 = \vartheta_{12}$.

For two arbitary directions, the probability of obtaining the same results is given by the Quantum Mechanics formula:

$$P(i+, j+) = \frac{1}{9} q_{\vartheta_{ij}} = \frac{1}{18}(1 - \cos\vartheta_{ij})$$

Then we subtitute to Bell's inequality

$$\frac{1}{18}(1 - \cos\vartheta_{12}) + \frac{1}{18}(1 - \cos\vartheta_{23}) \geq \frac{1}{18}(1 - \cos\vartheta_{13})$$

We found the function determining when Bell's inequalities hold and when they do not.

$$f(\vartheta_{12}, \vartheta_{23}) = 1 + \cos(\vartheta_{12} + \vartheta_{23}) - \cos\vartheta_{12} - \cos\vartheta_{23}$$

Analytically, we have found the minima and the maxima of the function.
The minima are at $[+\frac{\pi}{4}, +\frac{\pi}{4}]$ and $[-\frac{\pi}{4}, -\frac{\pi}{4}]$.
The maxima are at $[+\frac{\pi}{4}, -\frac{\pi}{4}]$ and $[-\frac{\pi}{4}, +\frac{\pi}{4}]$.
We found at which points the inequality is violated the most. This can be seen in figure 7.

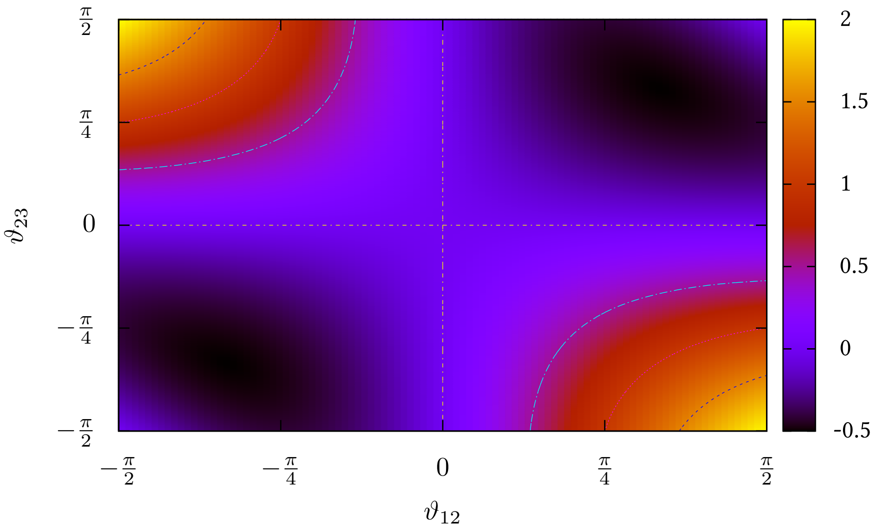

Figure 7: Degree of violation of Bell's inequality

5 Conclusions

In this miniproject we have proven that not every physical system is causal and that Bell's inequalities do not always hold. We have proven that Bell's inequalities always hold in a causal system.
The Bell Test of Quantum Mechanics has a great potential to be the keystone for quantum cryptography. The mechanism studied in this paper may be used to determine whether transported data was intercepted or not.

6 Acknowledgements

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7 References

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