Symmetry of Nature and Nature of Symmetry

Z. Grycová, M. Král, J. Kulhavý, B. Růžičková, B. Zemanová

FZÚ AV ČR, Na Slovance 1999/2 Praha 8
zuzanagrycova6@seznam.cz, martin.kral.03@gmail.com,
jakubkulhavy@pm.me, barar.mt@gmail.com, zemanova.be@gmail.com

June 21, 2022

Abstract

Action principle and its connection to symmetry has been at the heart of all modern theoretical physics. In this project we looked into the action for Newtonian point particles and examined the various symmetries it can possess and the corresponding conservation laws. In particular, we proved that Newtonian mechanics allows only one universal clock, because it has time-translation symmetry which leads to the conservation of energy. We conclude with a general discussion of the maths of symmetry and what role it plays in other areas of physics such as the Standard model.

1 Introduction

Our project is based on looking at the Newton's Laws of motion through a different lens. In a nutshell, Newton's laws tell us that the questions ´how a particle moves?´ and ´why a particle moves?´ are intimately related. Motion occurs because of external forces acting on an object which directly determines its acceleration. Despite the successes of Newton's laws, physicists came up with a new principle called '*The Principle of Least Action*'. This new principle, as it turns out, describes every branch of classical physics and is also at the heart of quantum mechanics.

2 Action Principle

The principle of least action asks the question, what is the path taken by a particle from a given point to another given point in two specific instances of time. To answer this, one defines a function for every possible path, called an *action* as follows

$$S[q(t)] = \int_{t_1}^{t_2} L\left[q(t), \dot{q}\right] dt \tag{1}$$

where $q(t)$ denotes the position, $\dot{q} = dq/dt$ is the velocity, and L is a specific function of position and velocity called a *Lagrangian.* Although both the action and the Lagrangian are purely abstract mathematical quantities and cannot be measured, we will see they can describe almost all laws of physics.

The principle says that the path followed by a classical particle will be such that the action is either at a minimum or maximum. In other words, the variation of the action about the classical path will equal zero, i.e.

$$\delta S = 0 \; . \tag{2}$$

This condition is equivalent to the following equation involving the Lagrangian (for derivation see [1])

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{3}$$

These are known as the Euler-Lagrange equations (also called the equations of motion).

Each area of physics has a corresponding Lagrangian and the Euler-Lagrange equations tell us how the system behaves. The Lagrangian that covers Newton's laws of motion is the difference between kinetic and potential energy of a system.

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q) \tag{4}$$

Unlike Newton's laws, the equations of motion can be used in any coordinate system without complications, which is especially useful for working in non-inertial coordinates without introducing fictitious forces.

Similarly, using suitable Lagrangians, the principle of least action can be applied to electromagnetism (giving rise to Maxwell's equations), gravity, weak force and strong force. Identifying the correct variables on which the Lagrangian depends and the form of the Lagrangian has been pivotal to all of theoretical physics since the last century. This is known as the action paradigm.

3 Symmetry and Conserved Quantities

We all are familiar with the notion of symmetry. For example, if we can rotate an object less than 360 degrees and still get the same looking object, we talk about rotational symmetry. Likewise we can find translation symmetry, reflection symmetry and many more.

In physics, symmetry has a similar meaning. Instead of objects, we move and rotate coordinate systems and other parameters of the theory. It is a transformation, either of the degrees of freedom or other parameters, such that the action under these transformations does not change. If the action remains the same under these transformations, then so will the equations of motion and therefore all other physical observables of the system. Knowing what symmetries our theory must have is crucial in writing down the correct Lagrangian for it.

One of the important consequences of symmetry in action is due to a famous result known as the Noether's theorem. It states that if a system has a continuous symmetry, then it also has a conserved quantity that can be determined from the symmetry transformations.

For example, if a system has translation symmetry, we can prove it must have conserved momentum. If a theory has rotational symmetry, we know for sure it necessarily conserves angular momentum.

It becomes more interesting if instead of changing the coordinates we look at the change of time. It is known that in relativistic physics every observer has their own clock ticking at a different rate than others. But how does this work in Newtonian mechanics? Does it only allow a single universal clock, or multiple?

Let us take two clocks, one measuring time as t and a second one as a function $\tau(t)$. The action in the t-clock would look like

$$S_t = \int dt \left[\frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}t} \right)^2 - V(q) \right] \tag{5}$$

whereas for the $\tau(t)$ clock we will have

$$S_{\tau(t)} = \int d\tau \left[\frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 - V(q) \right] \tag{6}$$

And after some adjustments, with the goal of making the action in eq.6 directly comparable to action of t, we can get the action for $\tau(t)$ to look like

$$S_{\tau(t)} = \int dt \left[\frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}t} \right)^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right) - V(q) \left(\frac{\mathrm{d}\tau}{\mathrm{d}t} \right) \right] \tag{7}$$

From our experience with motion of real particles, we know that there are not two different sets of equations describing its motion. Ergo, we conclude the actions of t and $\tau(t)$ must be equal. From eq.7 we can see that would be the case if and only if

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t} \right) = 1 \tag{8}$$

Therefore we know that $\tau(t) = t + \alpha$, where α is a constant. This result tells us that the Newtonian mechanics has only one universal clocks. We can also see that this system has a time translation symmetry - we are free to choose when our clock shows $t = 0$, but all clocks must tick at the same rate!

Because we have time translation symmetry, Noether's theorem demands we must have a corresponding conserved quantity. The conserved quantity is nothing but our familiar friend total energy. We just derived the law of conservation of energy ´Energy can neither be created nor destroyed - only converted from one form of energy to another´ and understood where does it come from!

4 Symmetry Groups

Group theory is a branch of mathematics that studies symmetry. All symmetry transformations share a set of common properties, which define a group as follows.
Group is a set of elements (G) along with an operation $(\cdot)$ that satisfies the following conditions:

1. For any two elements of set G, $g_1 \cdot g_2$ also belongs to G.
2. There exists an element such that $g \cdot e = e \cdot g = g$.
3. For each element there exists a unique g such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.
4. For any three elements, the following holds $g_1 \cdot [g_2 \cdot g_3] = [g_1 \cdot g_2] \cdot g_3$.

Set of all integers under addition is an example of group.

Matrix groups play a huge role in the standard model of particle physics. An example of matrix group is the set of all $n \times n$ orthogonal matrices - $O(n)$. The most important example is the set of all $n \times n$ unitary matrices with determinant equal to 1 - $SU(n)$.

5 Symmetry in Particle Physics

Every non-gravitational force in nature and its interactions with matter are described by a theory known as the standard model. Standard model stands tall above all other theories in all of science in being the most accurate. In the 1950's hundreds of new particles were discovered and it represented a crisis. The standard model as we know it today was realized only when physicists identified the correct symmetry groups for the three forces - $U(1)$ for Electromagnetism, $SU(2)$ for weak force, and $SU(3)$ for strong force. Once we understood the symmetry property of the forces it also explained how matter particles interacted with each other through these forces.

Standard model, despite its successes, is not the end of the story. But physicists still use action and symmetries to search for physics beyond the standard model.

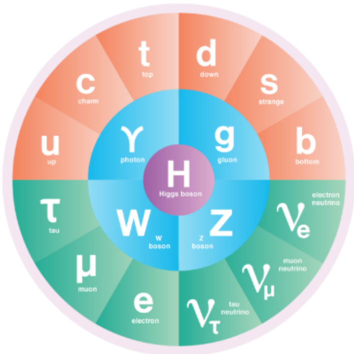

Figure 1: All of Standard Model particles

6 Conclusion

We saw that if time has no beginning or end, total energy is conserved. However, we now know that the universe started with the Big Bang about 13.8 billion years ago and therefore time has a beginning. Should not this then imply that total energy in our universe is not conserved? Exactly, it is not! A simple example that illustrates this is the observed red shift of distant galaxies. As the universe expands, the distance between the light source and the receiver increases. This causes the wavelength of light to stretch and distant objects seem more red to us. However, red shifted light has less energy than it had before. This means that a part of the energy had disappeared.

Acknowledgements

We wholeheartedly thank our mentor Subhroneel Chakrabarti, Ph.D. for his guidance. We would also like to thank the organizers of The Week of Science.

References

[1] D. J. Morin. *Introduction to Classical Mechanics: With Problems and Solutions*. Cambridge University Press, 2008.